Electronic Journal of Qualitative Theory of Differential Equations
2018, No. **66**, 1–19; https://doi.org/10.14232/ejqtde.2018.1.66 www.math.u-szeged.hu/ejqtde/

On the asymptotic stability for intermittently damped nonlinear oscillators

László Hatvani ✉

University of Szeged, Bolyai Institute, Aradi vértanúk tere 1, H-6720 Szeged, Hungary

Received 11 April 2018, appeared 30 July 2018
Communicated by Ferenc Hartung

Abstract. The second order nonlinear differential equation

$$x'' + h(t, x, x')x' + f(x) = 0 \qquad (x \in \mathbb{R},\ t \in \mathbb{R}_+ := [0, \infty),\ ()' := \tfrac{\mathrm{d}}{\mathrm{d}t}()),$$

and a sequence $\{I_n\}_{n=1}^{\infty}$ of non-overlapping intervals are given, where the damping coefficient h admits an estimate

$$a(t)|y|^{\alpha} w(x, y) \le h(t, x, y) \le b(t)W(x, y) \qquad (t \in I := \cup_{n=1}^{\infty} I_n; x, y \in \mathbb{R}).$$

It is known that if the equation is linear ($f(x) \equiv x$, $h(t, x, x') \equiv h(t)$, $a(t) \le h(t) \le b(t)$), $a(t) \ge \underline{a} = \text{const.} > 0$ and $b(t) \le \overline{b} = \text{const.} < \infty$, then $\sum_{n=1}^{\infty} |I_n|^3 = \infty$ is sufficient for the asymptotic stability, and the exponent 3 is the best possible. (Here $|I_n|$ denotes the length of I_n.) We give sufficient conditions for the asymptotic stability of the zero solution via the control functions a, b on the control set I considering cases when $\underline{a} > 0$ and/or $\overline{b} < \infty$ do not exist.

Keywords: intermittent damping, asymptotic stability, total mechanical energy, dissipation, differential inequalities.

2010 Mathematics Subject Classification: Primary 34D20, Secondary 70K20.

1 Introduction

We consider the model

$$x'' + h(t, x, x')x' + f(x) = 0 \qquad (x \in \mathbb{R},\ t \in \mathbb{R}_+ := [0, \infty),\ ()' := \tfrac{\mathrm{d}}{\mathrm{d}t}()), \tag{1.1}$$

of a nonlinear damped oscillator, where $-f(x)$ is the restoring force ($f : (-\overline{M}, \overline{M}) \to \mathbb{R}$ is continuous, $0 < \overline{M} \le \infty$ is a fixed constant, $xf(x) > 0$ if $x \ne 0$); $h : \mathbb{R}_+ \times (-\overline{M}, \overline{M}) \times \mathbb{R} \to \mathbb{R}_+$ is the damping coefficient, which allows an estimate

$$a(t)|y|^{\alpha} w(x, y) \le h(t, x, y) \le b(t)W(x, y). \tag{1.2}$$

✉ Email: hatvani@math.u-szeged.hu

Here α is a nonnegative real number; $w, W : (-\overline{M}, \overline{M}) \times \mathbb{R} \to \mathbb{R}_+$ are continuous, $w(x, y) > 0$ for all x, y. Functions $a, b : \mathbb{R}_+ \to \mathbb{R}_+$ are piecewise continuous, they are called the lower and the upper control function respectively: we can control the damping via these functions. *Intermittent damping* means that we are given a sequence $\{I_n = (\alpha_n, \beta_n)\}_{n=1}^{\infty}$ ($\lim_{n\to\infty} \alpha_n = \infty$) of non-overlapping intervals, and it is supposed that a, b can be controlled only over these intervals in time; between the intervals nothing but the nonnegativity of $a(t), b(t)$ is supposed. $I := \cup_{n=1}^{\infty} I_n$ will be called the *control set*. *The problem* is to find conditions on $a, b, \{I_n\}$ guaranteeing the asymptotic stability for the equilibrium state $x = x' = 0$ in Lyapunov's sense [3]. This is the problem of intermittent damping [5,6,8,10–13].

Let us introduce the following notations:

$$H(x,y) := \frac{1}{2}y^2 + F(x) \text{ (total mechanical energy)}, \quad F(x) := \int_0^x f\,;$$
$$H_M := \{(x,y) \in (-\overline{M}, \overline{M}) \times \mathbb{R} : H(x,y) < \min\{F(M), F(-M)\}\};$$
$$0 < K_M := \left(\sup\left\{\frac{f(x)}{x} : 0 < |x| \le M\right\}\right)^{1/2} \le \infty \quad (0 < M < \overline{M}); \tag{1.3}$$
$$A(t) := \int_0^t a(s)\,\mathrm{d}s, \quad B(t) := \int_0^t b(s)\,\mathrm{d}s;$$
$$\underline{a}_n := \inf\{a(t) : \alpha_n < t < \beta_n\} \quad \overline{a}_n := \sup\{a(t) : \alpha_n < t < \beta_n\};$$
$$A_n := \int_{\alpha_n}^{\beta_n} a(t)\,\mathrm{d}t, \quad B_n := \int_{\alpha_n}^{\beta_n} b(t)\,\mathrm{d}t \quad (n \in \mathbb{N});$$

$\underline{b}_n$, $\overline{b}_n$ are defined analogously with $\underline{a}_n$, $\overline{a}_n$. It is easy to see that for every $M \in (0, \overline{M})$ the closure of H_M is compact, and there is a constant $c(M) > 0$ such that

$$a(t) \le c(M)b(t) \qquad (t \in \mathbb{R}_+). \tag{1.4}$$

The derivative of H with respect to (1.1) [3] is

$$H'(t,x,x') = -h(t,x,x')(x')^2 \le 0, \tag{1.5}$$

so H is non-increasing along any solution of (1.1) (the energy is dissipated). Therefore H_M is an invariant neighborhood of $(0,0)$, so the equilibrium is stable. We always suppose that $(x(0), x'(0)) \in H_M$, so $|x(t)| < M$ is also satisfied for all $t \in \mathbb{R}_+$. It has remained to find conditions for the control functions a, b guaranteeing $\lim_{t\to\infty} H(t, x(t), x'(t)) = 0$ for every solution $t \mapsto x(t)$ with $(x(0), x'(0)) \in H_M$.

The first result on the intermittent damping was published by R. A. Smith [13] for the linear case

$$h(t,x,x') \equiv h(t), \quad f(x) \equiv x, \quad a(t) \equiv b(t) \equiv h(t). \tag{1.6}$$

Theorem A. *Suppose that*

$$\sum_{n=1}^{\infty} \underline{h}_n |I_n| \left(\min\left\{|I_n|; \frac{1}{1+\overline{h}_n}\right\}\right)^2 = \infty, \tag{1.7}$$

where $|I_n|$ denotes the length of I_n. Then the zero solution of (1.1) is asymptotically stable.

This theorem was improved in the linear case (1.6) in [10]. P. Pucci and J. Serrin [12] proved theorems for very general quasi-variational ordinary differential systems of many degrees of freedom. Here we cite the consequences of their two main theorems for (1.1) using the notation

$$d_n := \frac{1}{|I_n|} \int_{I_n} b$$

end the concept of integral positivity.

Definition 1.1. A locally integrable function $g : \mathbb{R}_+ \to \mathbb{R}_+$ is called *integrally positive with parameter* $\delta > 0$ if

$$\liminf_{t\to\infty} \int_t^{t+\delta} g > 0. \tag{1.8}$$

If (1.8) is satisfied for all $\delta > 0$, then g is called *integrally positive*.

Theorem B. *Suppose that there exist positive constants c_1, c_2, c_3 such that*

$$\sum_{n=1}^{\infty} \underline{a}_n |I_n| \min\left\{ |I_n|^{2+\alpha}; \frac{c_1}{c_2 + c_3 \underline{a}_n d_n} \right\} = \infty. \tag{1.9}$$

Then the zero solution of (1.1) *is asymptotically stable.*

Theorem C. *Suppose that the lower control function a is integrally positive and there exists a positive constant c_4 such that*

$$\sum_{n=1}^{\infty} \min\left\{ |I_n|^2; \frac{c_4}{d_n} |I_n| \right\} = \infty. \tag{1.10}$$

Then the zero solution of (1.1) *is asymptotically stable.*

These theorems yield the following rather unexpected corollaries for simpler control functions.

Corollary D. *I. If*

$$a(t) \geq \underline{a} > 0, \quad b(t) \leq \overline{b} < \infty \qquad (t \in I) \tag{1.11}$$

with some constants $\underline{a}$, $\overline{b}$, and

$$\sum_{n=1}^{\infty} |I_n|^{3+\alpha} = \infty, \tag{1.12}$$

then the zero solution of (1.1) *is asymptotically stable.*

II. If

$$a(t) \geq \underline{a} > 0 \quad (t \in \mathbb{R}_+), \qquad b(t) \leq \overline{b} < \infty \quad (t \in I),$$

and

$$\sum_{n=1}^{\infty} |I_n|^2 = \infty, \tag{1.13}$$

then the zero solution of (1.1) *is asymptotically stable.*

Pucci and Serrin [12] showed that the exponents $3 + \alpha$ and 2 in (1.12) and (1.13), respectively are the best possible ones in the sense that without further restrictions no smaller exponents can yield the general conclusion. For this reason we call Theorem B, respectively Theorem C, a result of "type exponent $3 + \alpha$", respectively of "type exponent 2".

In this paper we prove theorems of types $3 + \alpha$ and 2, which do not use the infimum $\underline{a}_n$, therefore they will be applicable when the lower control function a often vanishes even on intervals of a fixed length. Pucci and Serrin based their results on the method of quasi-variational inequalities. We use an entirely different approach, the method of differential inequalities.

The paper is organized as follows. In Section 2 and 3 we formulate the basic theorems and their corollaries and discuss the results. In Section 4 we give the proofs.

2 Results of type "exponent $3+\alpha$"

Equation (1.1) is equivalent to the system

$$x' = y, \quad y' = -f(x) - h(t,x,y)y. \tag{2.1}$$

By the use of the polar coordinates r, φ with

$$x = r\cos\varphi, \quad y = r\sin\varphi \qquad (r > 0, -\infty < \varphi < \infty), \tag{2.2}$$

this system can be rewritten into the form

$$r' = r\sin\varphi\cos\varphi - f(r\cos\varphi)\sin\varphi - h(t, r\cos\varphi, r\sin\varphi)\sin^2\varphi, \tag{2.3}$$

$$\varphi' = -\left(\sin^2\varphi + \frac{f(r\cos\varphi)}{r\cos\varphi}\cos^2\varphi\right) - h(t, r\cos\varphi, r\sin\varphi)\sin\varphi\cos\varphi. \tag{2.4}$$

In what follows, if $t \mapsto (x(t), y(t))$ is a solution of (2.1), then we mention the same solution as "solution $t \mapsto (r(t), \varphi(t))$", provided that $x(t), y(t)$ and $r(t), \varphi(t)$ are connected via (2.2).

For a solution $t \mapsto (x(t), y(t))$ starting from H_M $(M > 0)$ introduce the notations

$$h_*(t) := h(t, x(t), y(t)), \quad H_*(t) := H(x(t), y(t)). \tag{2.5}$$

The proof of the main result will be based upon the method of contradiction. We will suppose that the equilibrium is not asymptotically stable, i.e., there exists a point $(x(0), y(0)) \in H_M$ such that for the solution $(x(t), y(t))$ starting from this point there holds $H_*(\infty) := \lim_{t\to\infty} H_*(t) > 0$. Then it can be seen that

$$\liminf_{t\to\infty} r(t) =: r_0 > 0. \tag{2.6}$$

Integrating (1.5) we get a contradiction if we have the divergence

$$\begin{aligned} H_*(0) - H_*(\infty) &\geq r_0^2 \int_0^\infty h_*(t)\sin^2\varphi(t)\,\mathrm{d}t \\ &\geq r_0^{2+\alpha}\left(\inf_{H_M} w(x,y)\right)\int_0^\infty a(t)|\sin\varphi(t)|^{2+\alpha}\,\mathrm{d}t = \infty. \end{aligned} \tag{2.7}$$

However, we cannot require directly divergence (2.7) of $a(t)$ because we do not know $\varphi(t)$ from the solution (r, φ). The main idea is that we estimate $|\sin\varphi(t)|$ from below, the estimation defines an appropriate family of test functions on the control set I and in the theorem we require the divergence of the integral of the products of $a(t)$ and the test functions.

Theorem 2.1. I. *Suppose that for every $\gamma > 0$, for every sequence of non-overlapping intervals $\{I_n = (\alpha_n, \beta_n)\}_{n=1}^\infty$ of the property $\beta_n - \alpha_n \leq \gamma$, and for arbitrary $\xi_n \in I_n$ the divergence*

$$\begin{aligned} \sum_{n=1}^\infty \Bigg(\int_{\alpha_n}^{\xi_n} a(t)\left(\min\left\{\int_{\alpha_n}^t \exp[-q(B(t)-B(s))]\,\mathrm{d}s; \xi_n - t\right\}\right)^{2+\alpha}\mathrm{d}t \\ + \int_{\xi_n}^{\beta_n} a(t)\left(\min\left\{\int_{\xi_n}^t \exp[-q(B(t)-B(s))]\,\mathrm{d}s; \beta_n - t\right\}\right)^{2+\alpha}\mathrm{d}t\Bigg) \\ = \infty \qquad (q = q(M) := \sup\nolimits_{H_M} W(x,y)) \end{aligned} \tag{2.8}$$

holds. Then the equilibrium of (1.1) is asymptotically stable.

II. *Suppose $K_M < \infty$. If (2.8) holds for some $\gamma < \pi/K_M$, then the equilibrium of (1.1) is asymptotically stable.*

If we estimate $\int_{\alpha_n}^{t} \exp[-q(B(t) - B(s))]\,\mathrm{d}s$ in (2.8) in different ways, then we get different corollaries. At first we use $\overline{b}_n = \sup_{I_n} b$.

Corollary 2.2. *Suppose that there are a sequence $\{I_n\}$ of non-overlapping intervals and a number $\kappa \in (0,1)$ such that*

$$\sum_{n=1}^{\infty} \left(\min \left\{ |I_n|; \frac{1}{1+\overline{b}_n} \right\} \right)^{2+\alpha} \int_{E_n} a = \infty \tag{2.9}$$

holds for every sequence $\{E_n\}$ of sets $E_n \subset I_n$ such that E_n is the union of finite subintervals of I_n and $\mathrm{mes}(E_n) \geq \kappa |I_n|$*, where* $\mathrm{mes}(E_n)$ *denotes the Lebesgue measure of E_n. Then the equilibrium of* (1.1) *is asymptotically stable.*

This corollary is a generalization of Smith's Theorem A to the nonlinear equation (1.1). What is more, in the special case (1.6) it implies a sharpened version of Theorem A working also in the case $\underline{h}_n = 0$. We can get a more general result if we use $B_n = \int_{I_n} b$ instead of $\overline{b}_n$.

Corollary 2.3. *Suppose that for every $\gamma > 0$ there are a sequence $\{I_n\}$ of non-overlapping intervals with $|I_n| \leq \gamma$ and a number $\kappa \in (0,1)$ such that*

$$\sum_{n=1}^{\infty} \left(\exp\left[-q \int_{I_n} b \right] |I_n| \right)^{2+\alpha} \int_{E_n} a = \infty \tag{2.10}$$

holds for every sequence $\{E_n\}$ of sets $E_n \subset I_n$ such that E_n is the union of finite subintervals of I_n and $\mathrm{mes}(E_n) \geq \kappa |I_n|$*. Then the equilibrium of* (1.1) *is asymptotically stable.*

It is worth noticing that the condition $\kappa \in (0,1)$ in Corollaries 2.2 and 2.3 is sharp in the sense that if we require (2.9) and (2.10) with $E_n = I_n$ (the case of $\kappa = 1$), then the corollaries become false. In fact, e.g., if Corollary 2.2 were true with $\kappa = 1$, then $\int_0^\infty a = \infty$ (this is (2.9) with bounded b and $I_n = (n, n+1)$) would imply asymptotic stability for the zero solution of (1.1) provided b is bounded, but this is not true (see, e.g., [10, 12]). The following corollary gives a condition containing $\int_{I_n} a$ instead of $\int_{E_n} a$.

Corollary 2.4. *Suppose that for every $\gamma > 0$ there is a sequence $\{I_n\}$ of non-overlapping intervals with $|I_n| \leq \gamma$ such that*

$$\sum_{n=1}^{\infty} \left(\frac{1}{\overline{a}_n} \exp\left[-q \int_{I_n} b \right] \right)^{2+\alpha} \left(\int_{I_n} a \right)^{3+\alpha} = \infty \tag{2.11}$$

holds. Then the equilibrium of (1.1) *is asymptotically stable.*

Now we cite a result showing that Corollary 2.4, and consequently Theorem 2.1 are sharp enough. It is known that in the case of "small damping" ($0 \leq h(t) \equiv h(t,x,y) \leq \overline{h} < \infty$, $t \in \mathbb{R}_+$) there is no necessary and sufficient condition for the asymptotic stability even in the linear case except linear equations with step function coefficients. Á. Elbert [1] dealt with the linear equation

$$x'' + h(t)x' + x = 0, \quad h(t) := \begin{cases} h_n > 0, & \text{for } t \in I_n = [\omega_n, \omega_{n+1}),\ n \in \mathbb{N} \\ 0, & \text{otherwise.} \end{cases} \tag{2.12}$$

He solved (2.12) on intervals I_n, then, to get the global solution of an initial value problem, he "glued together" the pieces of the solution at the endpoints of I_n's so that the solution might be continuously differentiable on $[0,\infty)$. He proved the following

Theorem E. *Suppose that the sequences $\{|I_n|\}$ and $\{h_n|I_n|\}$ are bounded. Then the zero solution of* (2.12) *is asymptotically stable* if and only if

$$\sum_{n=1}^{\infty} h_n|I_n|^3 = \infty. \tag{2.13}$$

It is easy to see that under the conditions of Theorem 2.4 and Theorem E on $\{I_n\}$ and $\{h_n\}$ conditions (2.11) and (2.13) are equivalent. Therefore we can say that condition (2.11) is not only sufficient but also *necessary* in some sense.

Corollary D was unexpected because the effect of the damping was controlled from below via lengths $|I_n|$ in condition (1.12). Now we already see that this was made possible by the condition $a(t) \geq \underline{a} > 0$ in (1.11). One expects that in the general case without this condition it is $\int_{I_n} a$ that can be used for estimating the effect of damping from below (see (1.5)). Corollary 2.4 gives the possibility of dropping condition $a(t) \geq \underline{a} > 0$ from (1.11), and the result verifies this expectation.

Corollary 2.5. *Suppose that for every $\gamma > 0$ there is a sequence $\{I_n\}$ of non-overlapping intervals with $|I_n| \leq \gamma$ such that the sequence $\{\overline{b}_n\}$ is bounded and*

$$\sum_{n=1}^{\infty} \left(\int_{I_n} a \right)^{3+\alpha} = \infty \tag{2.14}$$

holds. Then the equilibrium of (1.1) *is asymptotically stable.*

Remark 2.6. For the sake of brevity, in Corollaries 2.3–2.5 we did not mention that in the case of $K_M < \infty$ the boundedness condition on $|I_n|$ can be weakened in the following way: it is enough to require that there exists a constant $\gamma \in (0, \pi/K_M)$ such that condition (2.10), (2.11), or (2.14) is satisfied for every sequence $\{I_n\}$ of the property $|I_n| \leq \gamma$.

In general, the boundedness condition on $|I_n|$ cannot be dropped from the corollaries. This can be seen very easily in the case of Corollary 2.5. Suppose that (2.14) alone guaranties asymptotic stability provided that $\{\overline{b}_n\}$ is bounded. However if there is no boundedness condition on $|I_n|$, then $\int_0^\infty a = \infty$ implies (2.14). In fact, it is enough to choose I_n so that $\int_{I_n} a \geq 1$. This would mean that $\int_0^\infty a = \infty$ guarantees asymptotic stability provided that $\{\overline{b}_n\}$ is bounded, but it is well-known that this is not true (see, e.g., [10,12]).

Fortunately, by the aid of a new method of proof we can strengthen condition (2.14) so that the boundedness condition on $|I_n|$ may be omitted.

Theorem 2.7. *Suppose that there is a sequence $\{I_k\}$ of non-overlapping intervals such that the sequence $\{\overline{b}_k\}$ is bounded and*

$$\sum_{k=1}^{\infty} \frac{1}{1+|I_k|^{2+\alpha}} \left(\int_{I_k} a \right)^{3+\alpha} = \infty \tag{2.15}$$

holds. Then the equilibrium of (1.1) *is asymptotically stable.*

3 Results of type "exponent 2"

In contrast with the previous one, in this section we always suppose that the damping is "large" in the sense that the lower control function is integrally positive with a suitable parameter not only on the control set I but *on the whole half-line* $[0, \infty)$. It can be proved [4] that under this condition there exists no oscillatory solution of the property $H_*(\infty) > 0$, so we have to deal with only non-oscillatory solutions. As is known, they are monotonous; we have to exclude the so-called *overdamping* $x(\infty) \neq 0$. R. A. Smith gave the first necessary and sufficient condition for the linear case (1.6) requiring

$$\int_0^\infty \int_0^t \exp\left[-\int_s^t h\right] \mathrm{d}s\, \mathrm{d}t = \infty \tag{3.1}$$

provided that the oscillator is controlled on the whole half-line $[0, \infty)$.

In this section we generalize (3.1) to the intermittent damping of nonlinear systems.

Theorem 3.1. *Suppose that*

$$\begin{cases} a \text{ is integrally positive} & \text{if } K_M = \infty, \\ a \text{ is integrally positive with parameter } \pi/K_M & \text{if } K_M < \infty. \end{cases} \tag{3.2}$$

If, in addition,

$$\sum_{n=1}^\infty \int_{\alpha_n}^{\beta_n} \left(\int_{\alpha_n}^t \exp[-(B(t) - B(s))]\, \mathrm{d}s \right) \mathrm{d}t = \infty, \tag{3.3}$$

then the zero solution of the equilibrium of (1.1) *is asymptotically stable.*

We give two more explicit corollaries using L_∞ and L_1 norm of b.

Corollary 3.2. *Suppose* (3.2)*. If, in addition,*

$$\sum_{n=1}^\infty \left(\frac{1}{\bar{b}_n} |I_n| - \frac{1}{\bar{b}_n^2} \left(1 - \exp[-\bar{b}_n |I_n|] \right) \right) = \infty, \tag{3.4}$$

then the zero solution of (1.1) *is asymptotically stable. Especially,* (3.2) *and*

$$\sum_{n=1}^\infty \frac{1}{\bar{b}_n} |I_n| = \infty \quad \text{and} \quad \sum_{n=1}^\infty \frac{1}{\bar{b}_n^2} < \infty \tag{3.5}$$

imply that the zero solution of (1.1) *is asymptotically stable.*

Corollary 3.3. *Suppose* (3.2)*. If, in addition,*

$$\sum_{n=1}^\infty \exp\left[-\int_{\alpha_n}^{\beta_n} b\right] |I_n|^2 = \infty, \tag{3.6}$$

then the zero solution of (1.1) *is asymptotically stable.*

Conditions (3.2)–(3.6) are knew; we can say that they extend (1.12)–(1.13) to unbounded control function b.

In comparison, e.g., with (1.10), condition (3.3) is not explicit enough for applications. Now we make it more explicit and applicable. Let us fix a constant $d > 0$ and introduce the notations

$$t_{n,k} := \inf\left\{ t \geq \alpha_n : \int_{\alpha_n}^t b = kd \right\},\ r_{n,k} := \min\{t_{n,k}; \beta_n\}, \quad (k = 0, 1, \dots). \tag{3.7}$$

Theorem 3.4. *For every $d > 0$ condition (3.3) is equivalent to*

$$\sum_{n=1}^{\infty}\left(\sum_{k=1}^{\infty}(r_{n,k}-r_{n,k-1})^2\right)=\infty. \tag{3.8}$$

Theorem C follows from Theorems 3.1 and 3.4 with $d = c_4$, namely condition (1.10) in Theorem C implies (3.8). In fact, if $B_n < c_4$, then $r_{n,1} - r_{n,0} = \beta_n - \alpha_n = |I_n| \geq |I_n|/\sqrt{2}$. If $B_n \geq c_4$, then let $k_n \geq 1$ denote the smallest natural number of the property $r_{n,k_n} = \beta_n$. By Cauchy's inequality we have

$$\begin{aligned}\sum_{k=1}^{\infty}(r_{n,k}-r_{n,k-1})^2 &= \sum_{k=1}^{k_n}(r_{n,k}-r_{n,k-1})^2\\ &\geq \frac{1}{k_n}\left(\sum_{k=1}^{k_n}(r_{n,k}-r_{n,k-1})\right)^2=\frac{|I_n|^2}{k_n}>|I_n|^2\frac{1}{1+\dfrac{B_n}{c_4}}\geq\frac{1}{2}|I_n|^2\frac{c_4}{B_n},\end{aligned}$$

which completes the proof.

In [8] it is proved by an example that (3.8) does not imply (1.10). This means that Theorem 3.1 does not follow from Theorem C.

4 Proofs

We will need an earlier lemma [4, Lemma 2.2] to estimate the distances of consecutive zeros of $\sin\varphi(t)$ for an oscillatory solution (r,φ).

Lemma 4.1. *Suppose $K_M < \infty$ and consider a solution (r,φ) $((x(0),x'(0)) \in H_M)$. Suppose, in addition, that $0 < \varepsilon < \pi/2$, and*

$$\varphi(T) = -k\pi - \varepsilon, \quad \varphi(S) = -(k+1)\pi + \varepsilon$$

with some $k \in \mathbb{N}$, $0 < T < S$.

Then there are $\mu(\varepsilon) > 0$, $\nu(\varepsilon) > 0$, independent of T, S such that

$$\lim_{\varepsilon\to 0+0}\nu(\varepsilon)=0,\qquad \int_T^S h_*(t)\sin^2\varphi(t)\,\mathrm{d}t \geq \mu(\varepsilon)\left(\frac{\pi}{K_M}-(S-T)-\nu(\varepsilon)\right). \tag{4.1}$$

The following lemma estimates the waste of energy between two zeros of $\sin\varphi(t)$ for an oscillatory solution.

Lemma 4.2. *Let us given an oscillatory solution of (1.1) with $H_*(\infty) := \lim_{t\to\infty} H_*(t) > 0$, and let us use the notation $r_0 := \liminf_{t\to\infty} r(t) > 0$. Denote by $\{\tau_n\}_{n=1}^{\infty}$ the increasing sequence of all zeros of $\sin\varphi(t)$. Then there are constants $c_5 = c_5(r_0, M) > 0$ and $q = q(M) > 0$, independent of n such that for every $n \in \mathbb{N}$ we have*

$$\begin{aligned}H_*(\tau_n)-H_*(\tau_{n+1}) &\geq r_0^2\int_{\tau_n}^{\tau_{n+1}} h_*(t)\sin^2\varphi(t)\,\mathrm{d}t\\ &\geq c_5\int_{\tau_n}^{\tau_{n+1}} a(t)\left(\min\left\{\int_{\tau_n}^{t}\exp[-q(B(t)-B(s))\,\mathrm{d}s];\tau_{n+1}-t\right\}\right)^{2+\alpha}\mathrm{d}t.\end{aligned} \tag{4.2}$$

Proof. Suppose that $\varphi(\tau_n) = -k\pi$ and, consequently $\varphi(\tau_{n+1}) = -(k+1)\pi$ ($k \in \mathbb{N}$). Let $\tau_n' \in (\tau_n, \tau_{n+1})$ denote the first time point where $\varphi(\tau_n') = -k\pi - \pi/2$ holds. From (2.4) it follows that

$$(\varphi(t) - \varphi(\tau_n))' \le -p(r_0, M) - qb(t)(\varphi(t) - \varphi(\tau_n)) \qquad (\tau_n \le t \le \tau_n'), \tag{4.3}$$

$$q = q(M) := \sup_{H_M} W(x, y),$$

$$p(r_0, M) := \min\left\{\sin^2\varphi + \frac{f(x)}{x}\cos^2\varphi : (x, y) \in H_M, r \ge r_0\right\} > 0.$$

By the basic theorem of the theory of differential inequalities [3, Theorem III.4.1, p. 26] we have

$$\varphi(t) \le -k\pi - p(r_0, M)\int_{\tau_n}^t \exp[-q(B(t) - B(s))]\,\mathrm{d}s \qquad (\tau_n \le t \le \tau_n'). \tag{4.4}$$

On the other hand, $\varphi'(t) \le -p(r_0, M)$ on the interval $[\tau_n', \tau_{n+1}]$, therefore

$$\varphi(t) \ge -(k+1)\pi + p(r_0, M)(\tau_{n+1} - t) \qquad (\tau_n' \le t \le \tau_{n+1}). \tag{4.5}$$

Combining (4.4) and (4.5) we obtain

$$|\sin\varphi(t)| \ge \frac{2}{\pi}p(r_0, M)\min\left\{\int_{\tau_n}^t \exp[-q(B(t) - B(s))]\,\mathrm{d}s; \tau_{n+1} - t\right\} \qquad (\tau_n \le t \le \tau_{n+1}). \tag{4.6}$$

From this estimate and (1.5) we get (4.2) with

$$c_5 := \left(\inf_{H_M} w(x, y)\right)\left(\frac{2}{\pi}r_0 p(r_0, M)\right)^{2+\alpha}.$$

□

We also need an analogous lemma for non-oscillatory solutions.

Lemma 4.3. *For every non-oscillatory solution of* (1.1) *with* $H_*(\infty) > 0$ *there exists a* $T_* > 0$ *such that for arbitrary* τ_1, τ_2 ($T_* < \tau_1 < \tau_2$) *we have*

$$H_*(\tau_1) - H_*(\tau_2) \ge c_5 \int_{\tau_1}^{\tau_2} a(t)\left(\int_{\tau_1}^t \exp[-q(B(t) - B(s))\,\mathrm{d}s]\right)^{2+\alpha}\mathrm{d}t. \tag{4.7}$$

Proof. If (x, y) is a non-oscillatory solution, then it can bee seen that $x(t)y(t) < 0$ for t large enough, let us say if $t \ge T_*$, and $\lim_{t\to\infty}\varphi(t) \equiv 0 \pmod{\pi}$. Then

$$-\left(k + \frac{1}{4}\right)\pi < \varphi(t) < -k\pi \quad \text{for some } k \in \mathbb{Z}_+.$$

Similarly to (4.3) we obtain

$$(\varphi(t) + k\pi)' \le -p_{r_0,M} - qb(t)(\varphi(t) + k\pi) \qquad (T_* \le \tau^1 \le t \le \tau^2),$$

and

$$\begin{aligned}
\varphi(t) &\le -k\pi + (\varphi(\tau^1) + k\pi)e^{-q(B(t) - B(\tau_1))} - p_{r_0,M}\int_{\tau^1}^t e^{-q(B(t)-B(s))}\,\mathrm{d}s \\
&\le -k\pi - p_{r_0,M}\int_{\tau^1}^t e^{-q(B(t)-B(s))}\,\mathrm{d}s,
\end{aligned}$$

$$|\sin\varphi(t)| \ge \frac{2}{\pi}|\varphi(t) + k\pi| \qquad (\tau^1 \le t \le \tau^2),$$

from which (4.7) follows. □

Proof of Theorem 2.1. Suppose the contrary and let (x, y) be such a solution that $H_*(\infty) > 0$. If this solution is oscillatory, then let $\{\tau_n\}_{n=1}^{\infty}$ denote the increasing sequence of all zeros of $\sin\varphi(t)$. For a fixed n there is a $k \in \mathbb{Z}_+$ such that $\varphi(\tau_n) = -k\pi$ and $\varphi(\tau_{n+1}) = -(k+1)\pi$. Let $t_n, s_n \in (\tau_n, \tau_{n+1})$ denote the time points when $\varphi(t_n) = -k\pi - \pi/8$ and $\varphi(s_n) = -k\pi - 3\pi/8$ for the first time. Then

$$\varphi'(t) \geq -p(r_0, M) \qquad (t_n \leq t \leq s_n),$$

consequently,

$$\tau_{n+1} - \tau_n \geq s_n - t_n \geq \pi/(4p(r_0, M)). \tag{4.8}$$

We state that I_m contains at most one member of $\{\tau_n\}$ provided that m is large enough.

In fact, at first let us consider the case $K_M = \infty$, choose a γ from the interval $(0, \pi/(4p(r_0, M)))$ and an $\{I_m\}$ with $|I_m| \leq \gamma$ and such that (2.8) is satisfied. In view of (4.8) the statement is true for arbitrary m. If $K_M < \infty$, then choose $\gamma \in (0, \pi/K_M)$ and $\{I_m\}$ so that $|I_m| \leq \gamma$ and (2.8) hold, and suppose that the statement is not true. Since $|I_m| \leq \gamma$ for all $m \in \mathbb{N}$, this means that there are infinitely many n's with $\tau_{n+1} - \tau_n \leq \gamma < \pi/K_M$. Let us denote by $\{(\tau'_k, \tau'_{k+1})\}_{k=1}^{\infty}$ the subsequence of $\{(\tau_n, \tau_{n+1})\}$ with this property. Choosing $\varepsilon > 0$ applying so small that $\nu(\varepsilon) < ((\pi/K_M) - \gamma)/2$ and applying (2.7) and Lemma 4.1 we obtain

$$\begin{aligned} H_*(0) - H_*(\infty) &\geq r_0^2 \sum_{m=1}^{\infty} \int_{\tau'_m}^{\tau'_{m+1}} h_*(t) \sin^2\varphi(t)\,\mathrm{d}t \\ &\geq r_0^2 \mu(\varepsilon) \sum_{m=1}^{\infty} \left(\left(\frac{\pi}{K_M} - \gamma \right) - \nu(\varepsilon) \right) = \infty, \end{aligned}$$

which is a contradiction.

We have proved that I_m contains at most one member of $\{\tau_n\}$ provided that m is large enough. Without loss of the generality we may drop the finitely many I_m's containing at least two τ_n's.

Now we estimate the integrals over $[\tau_n, \tau_{n+1}]$ in (2.2) by integrals on I_m. We will use the simple fact that if $\tau < \alpha < t$, then

$$\int_{\tau}^{t} \exp[-q(B(t) - B(s))\,\mathrm{d}s] \geq \int_{\alpha}^{t} \exp[-q(B(t) - B(s))\,\mathrm{d}s].$$

We consider only those (τ_n, τ_{n+1})'s that have points from the control set $I = \cup_{m=1}^{\infty} I_m$: $n_1 < n_2 < \cdots < n_k < \cdots$ are the natural numbers such that $(\tau_{n_k}, \tau_{n_k+1}) \cap I \neq \emptyset$. Let us fix a $k \in \mathbb{N}$ and denote by $I_{p_k}, I_{p_k+1}, \ldots, I_{q_k}$ $(1 \leq p_k \leq q_k \leq p_{k+1})$ the control intervals having common points with $(\tau_{n_k}, \tau_{n_k+1})$:

$$I_{p_k+j} \cap (\tau_{n_k}, \tau_{n_k+1}) \neq \emptyset \quad (j = 0, 1, \ldots, q_k - p_k).$$

Then we have the estimate

$$\begin{aligned} &H_*(\tau_{n_k-1}) - H_*(\tau_{n_k+2})) \\ &\geq \sum_{j=0}^{q_k-p_k} c_5 \left(\int_{\alpha_{p_k+j}}^{\xi_{p_k+j}} a(t) \left(\min\left\{ \int_{\alpha_{p_k+j}}^{t} e^{-q(B(t)-B(s))\,\mathrm{d}s}; \xi_{p_k+j} - t \right\} \right)^{2+\alpha} \mathrm{d}t \right. \\ &\quad \left. + \int_{\xi_{p_k+j}}^{\beta_{p_k+j}} a(t) \left(\min\left\{ \int_{\xi_{p_k+j}}^{t} e^{-q(B(t)-B(s))\,\mathrm{d}s}; \beta_{p_k+j} - t \right\} \right)^{2+\alpha} \mathrm{d}t \right). \end{aligned} \tag{4.9}$$

If $\tau_{n_k} \in I_{p_k}$ ($\tau_{n_{k+1}} \in I_{q_k}$), then $\xi_{p_k} = \tau_{n_k}$ ($\xi_{q_k} = \tau_{n_{k+1}}$), otherwise ξ_{p_k+j} is arbitrary in I_{p_k+j}. On the other hand

$$\begin{aligned} 3(H_*(0) - H_*(\infty)) &\geq 3\left(\sum_{n=1}^{\infty}(H_*(\tau_n) - H_*(\tau_{n-1}))\right) \\ &\geq 3\left(\sum_{k=1}^{\infty}(H_*(\tau_{n_k+2}) - H_*(\tau_{n_k-1}))\right). \end{aligned} \tag{4.10}$$

It follows from (4.9) and (4.10) that condition (2.8) implies $H_*(0) - H_*(\infty) = \infty$, that is a contradiction.

If the solution (x, y) is non-oscillatory, then we apply Lemma 4.3. There exists a natural number m_* such that $m > m_*$ implies $\alpha_m > T_*$, so from (4.7) we obtain

$$\begin{aligned} H_*(T_*) - H_*(\infty) &\geq -\int_{[T_*,\infty)\cap I} H_*'(t)\,\mathrm{d}t \\ &\geq c_5 \sum_{m=m_*}^{\infty} \int_{\alpha_m}^{\beta_m} a(t)\left(\int_{\alpha_m}^{t} \exp[-q(B(t) - B(s))\,\mathrm{d}s]\right)^{2+\alpha} \mathrm{d}t \\ &\geq c_5 \sum_{m=m_*}^{\infty} \int_{\alpha_m}^{\beta_m} a(t)\left(\min\left\{\int_{\alpha_m}^{t} \exp[-q(B(t) - B(s))\,\mathrm{d}s]; \beta_m - t\right\}\right)^{2+\alpha} \mathrm{d}t. \end{aligned}$$

Condition (2.8) implies $H_*(0) - H_*(\infty) = \infty$ again. □

In what follows we will use the notation

$$\bar{g}_G := \sup\{g(t) : t \in G\} \qquad (G \subset \mathbb{R}_+, g : \mathbb{R}_+ \to \mathbb{R}).$$

Lemma 4.4.

$$\int_{\alpha}^{t} \exp[-q(B(t) - B(s))\,\mathrm{d}s] \geq \frac{1}{3}(t - \alpha) \qquad \left(\alpha \leq t \leq \frac{1}{q\bar{b}_{(\alpha,\beta)}}\right). \tag{4.11}$$

Proof. Since the function $t \mapsto 1 - \exp[-q\bar{b}_{(\alpha,\beta)}(t - \alpha)]$ is concave we have the estimate

$$\begin{aligned} \int_{\alpha}^{t} \exp[-q(B(t) - B(s))\,\mathrm{d}s] &\geq \int_{\alpha}^{t} \exp[-q\bar{b}_{(\alpha,\beta)}(t - s)]\,\mathrm{d}s \\ &= \frac{1}{q\bar{b}_{(\alpha,\beta)}}(1 - \exp[-q\bar{b}_{(\alpha,\beta)}(t - \alpha)]) \geq \frac{1}{3}(t - \alpha) \quad \left(\alpha \leq t \leq \alpha + \frac{\ln 3}{q\bar{b}_{(\alpha,\beta)}}\right), \end{aligned}$$

from which (4.11) follows. □

Proof of Corollary 2.2. Suppose that there exist $\{I_n\}$ and $\kappa \in (0, 1)$ such that condition (2.9) in Corollary 2.2 is satisfied. We show that condition (2.8) in Theorem 2.1 is also satisfied for the same $\{I_n\}$.

Starting from (2.8) and using the estimate (4.11) in Lemma 4.4 we obtain

$$\begin{aligned}
J_n^1 &:= \int_{\alpha_n}^{\xi_n} a(t)\left(\min\left\{\int_{\alpha_n}^{t}\exp[-q(B(t)-B(s))]\,\mathrm{d}s;\xi_n-t\right\}\right)^{2+\alpha}\mathrm{d}t\\
&\geq \int_{\alpha_n}^{\min\{\alpha_n+1/q\bar{b}_n;\xi_n\}} a(t)\left(\min\left\{\frac{1}{3}(t-\alpha_n;\xi_n-t)\right\}\right)^{2+\alpha}\mathrm{d}t\\
&\quad+\int_{\min\{\alpha_n+1/q\bar{b}_n;\xi_n\}}^{\xi_n} a(t)\left(\min\left\{\frac{1}{q\bar{b}_n}\left(1-\exp[-1]\right);\xi_n-t\right\}\right)^{2+\alpha}\mathrm{d}t\\
&\geq c_6\int_{\alpha_n}^{\xi_n} a(t)\left(\min\left\{t-\alpha_n;\xi_n-t;\frac{1}{1+\bar{b}_n}\right\}\right)^{2+\alpha}\mathrm{d}t
\end{aligned}$$

with a suitably chosen constant $c_6>0$ independent of n. Similarly,

$$\begin{aligned}
J_n^2 &:= \int_{\xi_n}^{\beta_n} a(t)\left(\min\left\{\int_{\xi_n}^{t}\exp[-q(B(t)-B(s))]\,\mathrm{d}s;\beta_n-t\right\}\right)^{2+\alpha}\mathrm{d}t\\
&\geq c_7\int_{\xi_n}^{\beta_n} a(t)\left(\min\left\{t-\xi_n;\beta_n-t;\frac{1}{1+\bar{b}_n}\right\}\right)^{2+\alpha}\mathrm{d}t
\end{aligned}$$

with a constant $c_7>0$ independent of n. Therefore there is a constant $c_8>0$ such that

$$\sum_{n=1}^{\infty}(J_n^1+J_n^2)\geq c_8\sum_{n=1}^{\infty}\int_{\alpha_n}^{\beta_n} a(t)\left(\min\left\{t-\alpha_n;|\xi_n-t|;\beta_n-t;\frac{1}{1+\bar{b}_n}\right\}\right)^{2+\alpha}\mathrm{d}t. \tag{4.12}$$

Now we choose a small $\delta_n>0$ (it will be defined later) and cut out δ_n-neighborhoods of α_n, ξ_n, and β_n. Then we get

$$\begin{aligned}
&\int_{\alpha_n}^{\beta_n} a(t)\left(\min\left\{t-\alpha_n;|\xi_n-t|;\beta_n-t;\frac{1}{1+\bar{b}_n}\right\}\right)^{2+\alpha}\mathrm{d}t\\
&\qquad\geq\left(\min\left\{\delta_n;\frac{1}{1+\bar{b}_n}\right\}\right)^{2+\alpha}\int_{E_n} a,
\end{aligned}\tag{4.13}$$

where

$$E_n := I_n\setminus\left([\alpha_n,\alpha_n+\delta_n]\cup[\xi_n-\delta_n,\xi_n+\delta_n]\cup[\beta_n-\delta_n,\beta_n]\right). \tag{4.14}$$

Let δ_n be defined by

$$\delta_n := ((1-\kappa)/4)|I_n|. \tag{4.15}$$

Combining (4.12), (4.13), and (2.9) we obtain

$$\sum_{n=1}^{\infty}(J_n^1+J_n^2)\geq c_8\left(\frac{1-\kappa}{4}\right)^{2+\alpha}\sum_{n=1}^{\infty}\left(\min\left\{|I_n|;\frac{1}{1+\bar{b}_n}\right\}\right)^{2+\alpha}\int_{E_n} a=\infty, \tag{4.16}$$

which means that divergence (2.8) is also satisfied.

It has remained to prove that for every $\gamma>0$ we may suppose that $|I_n|\leq\gamma$. This means that if (2.9) is satisfied for $\{I_n\}$ and κ, then it also holds for another sequence $\{L_m\}_{m=1}^{\infty}$ with the same κ, but $|L_m|\leq\gamma$ ($m\in\mathbb{N}$). The same fact was proved for linear systems in [7, Corollary 4.2]. To make the present paper self-contained we repeat the short proof here.

Let us observe at first the obvious fact that for arbitrary $\{u_n, v_n, w_n\}$ ($0<u_n<1$, $v_n, w_n>0$) and $\delta>0$ the two divergences

$$\sum_{n=1}^{\infty}\min\{u_n;v_n\}w_n=\infty,\qquad \sum_{n=1}^{\infty}\min\{u_n;v_n;\delta\}w_n=\infty$$

are equivalent.

If $|I_n| > \gamma$, then we divide the interval I_n into non-overlapping subintervals

$$I_n = \cup_{j=1}^{l_n} I_{nj}, \qquad \gamma/2 \le |I_{nj}| \le \gamma,$$

and take arbitrary sets $E_{nj} \subset I_{nj}$ which are finite union of intervals such that $\mathrm{mes}(E_nj) \ge \kappa |I_nj|$. Then

$$\sum_{j=1}^{l_n} \left(\min \left\{ \frac{1}{1+\bar{b}_{nj}}; |I_{nj}| \right\} \right)^{2+\alpha} \int_{E_{nj}} a \ge \left(\min \left\{ |I_n|; \frac{1}{1+\bar{b}_n}; \frac{\gamma}{2} \right\} \right)^{2+\alpha} \int_{E_n} a,$$

provided that $E_n := \cup_{j=1}^{l_n}$ We obtain $\{L_m\}$ if we exchange all I_n of the properties $|I_n| > \gamma$ with $\{I_{nj}\}_{j=1}^{l_n}$. □

Proof of Corollary 2.3. If we use the lower estimate

$$\int_{\alpha_n}^{t} \exp\left[-q(B(t) - B(s))\right] \mathrm{d}s \ge \exp\left[- \int_{\alpha_n}^{\beta_n} q(B(t) - B(s))\, \mathrm{d}s \right] (t - \alpha_n)$$

and its analogy on $[\xi_n, \beta_n]$, then for (2.8) we get an inequality analogous with (4.12):

$$\sum_{n=1}^{\infty} (J_n^1 + J_n^2) \ge c_9 \sum_{n=1}^{\infty} (\exp[-qB_n])^{2+\alpha} \int_{\alpha_n}^{\beta_n} a(t) \left(\min\left\{t - \alpha_n; |\xi_n - t|; \beta_n - t\right\}\right)^{2+\alpha} \mathrm{d}t. \tag{4.17}$$

Defining δ_n, E_n by (4.14), (4.15) we obtain

$$\sum_{n=1}^{\infty} (J_n^1 + J_n^2) \ge c_9 \left(\frac{1-\kappa}{4} \right)^{2+\alpha} \sum_{n=1}^{\infty} (\exp[-qB_n]|I_n|)^{2+\alpha} \int_{E_n} a = \infty, \tag{4.18}$$

which means that divergence (2.8) is also satisfied. □

Proof of Corollary 2.4. We start from (4.17). If we cut out δ_n-neighborhoods of α_n, ξ_n, and β_n, then we have

$$\begin{aligned} \sum_{n=1}^{\infty} (J_n^1 + J_n^2) &\ge c_9 \sum_{n=1}^{\infty} (\exp[-qB_n]\delta_n)^{2+\alpha} \int_{E_n} a \\ &\ge c_9 \sum_{n=1}^{\infty} (\exp[-qB_n]\delta_n)^{2+\alpha} \left(\int_{I_n} a - 4\delta_n \bar{a}_n \right). \end{aligned}$$

Defining $\delta_n := (1/8) \int_{I_n} a/\bar{a}_n$, from condition (2.11) we obtain

$$\sum_{n=1}^{\infty} (J_n^1 + J_n^2) \ge \frac{1}{2} \left(\frac{1}{8} \right)^{2+\alpha} c_9 \sum_{n=1}^{\infty} \left(\frac{\exp[-qB_n]}{\bar{a}_n} \right)^{2+\alpha} \left(\int_{I_n} a \right)^{3+\alpha} = \infty,$$

i.e., divergence (2.8) holds. □

Proof of Corollary 2.5. If we take into account (1.4), then (2.14) implies (2.11). □

Lemma 4.5. *Suppose that for every $\gamma > 0$ small enough and for every $\{\tau_n\}$ with $\gamma \le \tau_{n+1} - \tau_n \le 2\gamma$ we have*

$$\sum_{n=1}^{\infty} \int_{\tau_n}^{\tau_{n+1}} a(t) \left(\min \left\{ \int_{\tau_n}^{t} \exp[-q(B(t) - B(s))\, \mathrm{d}s]; \tau_{n+1} - t \right\} \right)^{2+\alpha} \mathrm{d}t = \infty. \tag{4.19}$$

Then the equilibrium of (1.1) is asymptotically stable.

Proof. Suppose the contrary and consider a solution (x, y) with $H_*(\infty) > 0$. If x is oscillatory, then denote by $\{\tau_n\}$ the increasing sequence of the zeros of $\sin\varphi(t)$. From (4.8) we know that $\tau_{n+1} - \tau_n \geq \pi/(4p(r_0, M))$. Take a $\gamma \in (0, \pi/(4p(r_0, M)))$. If $\tau_{n+1} - \tau_n \leq 2\gamma$ ($n \in \mathbb{N}$) is also satisfied, then by Lemma 4.2 condition (4.19) implies a contradiction. Now we show that we can always suppose $\tau_{n+1} - \tau_n \leq 2\gamma$ ($n \in \mathbb{N}$) in property (4.2).

In fact, if $\tau_{m+1} - \tau_m > 2\gamma$ for some natural number m, then add new members

$$\tau_{m,1} := \tau_m + \gamma,\ \tau_{m,2} := \tau_m + 2\gamma, \ldots, \tau_{m,j_m} := \tau_m + j_m\gamma \qquad (\tau_{m,0} := \tau_m,\ \tau_{m,j_m+1} := \tau_{m+1})$$

to the sequence $\{\tau_n\}$ as far as $\tau_{m+1} - (\tau_m + j_m\gamma) \leq 2\gamma$. After we have made this addition for all m of the property $\tau_{m+1} - \tau_m > 2\gamma$, we denote by $\{\tau'_n\}$ the new sequence. Firstly, $\tau'_{n+1} - \tau'_n) \geq \gamma$ for all n. Secondly, $\tau'_{n+1} - \tau'_n \leq 2\gamma$ ($n \in \mathbb{N}$). Finally, if (4.2) is satisfied for $\{\tau_m\}$ then (4.2) is also satisfied with $\{\tau'_m\}$ instead of $\{\tau_m\}$. In fact, obviously

$$\begin{aligned}\int_{\tau_m}^{\tau_{m+1}} a(t) \left(\min\left\{ \int_{\tau_m}^{t} e^{-q(B(t)-B(s))}\, ds; \tau_{m+1} - t \right\} \right)^{2+\alpha} dt \\ \geq \sum_{j=0}^{j_m} \int_{\tau_{m,j}}^{\tau_{m,j+1}} a(t) \left(\min\left\{ \int_{\tau_{m,j}}^{t} e^{-q(B(t)-B(s))}\, ds; \tau_{m,j+1} - t \right\} \right)^{2+\alpha} dt.\end{aligned}$$

This means that condition (4.19) always implies a contradiction via (4.2).

If x is non-oscillatory, then for arbitrary $\gamma > 0$ define an increasing sequence $\{\tau_n\}$ of the properties

$$\tau_1 > T_*, \quad \gamma \leq \tau_{n+1} - \tau_n \leq 2\gamma \qquad (n \in \mathbb{N}),$$

where T_* is taken from Lemma 4.3. Now condition (4.19) implies a contradiction by (4.7). $\square$

In the proofs of Corollaries 2.2–2.4 we established the method of cutting subintervals out. In the proof of Theorem 2.7 we have to apply it to a variety of intervals, so we formulate its variant we need into a lemma.

Lemma 4.6 (Cutting Out Lemma). *For arbitrary ξ, η ($0 \leq \xi < \eta < \infty$) we have*

$$\begin{aligned} J(\xi, \eta) &:= \int_{\xi}^{\eta} a(t) \left(\min\left\{ \int_{\xi}^{t} \exp[-q(B(t) - B(s))\, ds]; \eta - t \right\} \right)^{2+\alpha} dt \\ &\geq c_{10} \left(\int_{\xi}^{\eta} a \right)^{3+\alpha}, \end{aligned} \tag{4.20}$$

$$c_{10} = c_{10}(\xi, \eta) := \frac{1}{2} \left(\frac{1}{4c(M)\bar{b}_{(\xi,\eta)}} \right)^{2+\alpha} \exp[-(2+\alpha)q\bar{b}_{(\xi,\eta)}(\eta - \xi)].$$

Proof. Obviously we have

$$J(\xi, \eta) \geq \exp[-q(2+\alpha)\bar{b}_{(\xi,\eta)}(\eta - \xi)] \int_{\xi}^{\eta} a(t) \left(\min\{t - \xi; \eta - t\}\right)^{2+\alpha}.$$

Let $\delta > 0$ be small enough (it will be defined later). We cut out δ-neighborhoods of ξ and η:

$$\int_{\xi}^{\eta} a(t) \left(\min\{t - \xi; \eta - t\}\right)^{2+\alpha} \geq \delta^{2+\alpha} \int_{\xi+\delta}^{\eta-\delta} a \geq \delta^{2+\alpha} \left(\int_{\xi}^{\eta} a - 2\delta\bar{a}_{(\xi,\eta)} \right).$$

Choosing

$$\delta := \frac{1}{4\bar{a}_{(\xi,\eta)}} \int_{\xi}^{\eta} a$$

and taking into account also (1.4) we obtain (4.20). $\square$

Proof of Theorem 2.7. We want to use Lemma 4.5. Suppose that we have a sequence $\{I_k\}$ of property (2.15), and $\gamma > 0$, $\{\tau_n\}$ are arbitrary with $\gamma \le \tau_{n+1} - \tau_n \le 2\gamma$ ($n \in \mathbb{N}$). We show that (4.19) is satisfied.

Step 1. We modify $\{I_k\}$ a little bit to make it more treatable. If an interval I_p does not contain any member of $\{\tau_n\}$, then we do not change I_p. Otherwise let τ_j, respectively τ_m be the minimal, respectively the maximal member of the sequence $\{\tau_n\}$ in I_p. Consider the intervals (α_p, τ_j), (τ_j, τ_m), (τ_m, β_p) and denote by I'_p that one among them on which the integral of a is maximal. Let us exchange I_p for I'_p (and denote it by I_p again) in the sequence $\{I_k\}$. If the old sequence satisfied condition (2.15), then the new sequence of intervals also satisfies the same condition because of $\int_{I'_k} a \ge (1/3) \int_{I_k} a$.

Step 2. We make a classification for $\{I_k\}$. Taking an interval $I_k = (\alpha_k, \beta_k)$ arbitrarily, we have exactly two possibilities:

a) I_k is the union of more than one but finitely many members of $\{(\tau_n, \tau_{n+1})\}$:

$$I_k = (\tau_{m_k}, \tau_{m_k+1}) \cup (\tau_{m_k+1}, \tau_{m_k+2}) \cup \cdots \cup (\tau_{m_k+J_k}, \tau_{m_k+J_k+1}),$$

$(m_k \ge 1, J_k \ge 1)$;

b) there is an $n \in \mathbb{N}$ such that $I_k \subset (\tau_n, \tau_{n+1})$. Let $p_n, p_{n+1}, \ldots, p_n + P_n$ be all the indices possessing the properties $I_{p_n+p} \subset (\tau_n, \tau_{n+1})$ $(p = 0, 1, \ldots, P_n)$.

Step 3. We consider case a) and apply Cutting Out Lemma. Then we obtain

$$\begin{aligned}
L_j^k &:= \int_{\tau_{m_k+j}}^{\tau_{m_k+j+1}} a(t) \left(\min \left\{ \int_{\tau_{m_k+j}}^{t} \exp[-q(B(t) \; - B(s))] \, \mathrm{d}s; \tau_{m_k+j+1} - t \right\} \right)^{2+\alpha} \mathrm{d}t \\
&\ge c_{11} \left(\int_{\tau_{m_k+j}}^{\tau_{m_k+j+1}} a \right)^{3+\alpha}, \\
c_{11} &:= \frac{1}{2} \left(\frac{1}{4c(M)\bar{b}_I} \right)^{2+\alpha} \exp[-2\gamma(2+\alpha)q\bar{b}_I].
\end{aligned}$$

Step 4. By the power mean inequality [2, Theorem 86] and the property $0 < \gamma \le \tau_{n+1} - \tau_n$ we have

$$\begin{aligned}
\sum_{j=0}^{J_k} L_j^k &\ge c_{11} \sum_{j=0}^{J_k} \left(\int_{\tau_{m_k+j}}^{\tau_{m_k+j+1}} a \right)^{3+\alpha} \\
&\ge c_{11} \frac{1}{(J_k+1)^{2+\alpha}} \left(\sum_{j=0}^{J_k} \int_{\tau_{m_k+j}}^{\tau_{m_k+j+1}} a \right)^{3+\alpha} \ge c_{11} \gamma^{2+\alpha} \frac{1}{|I_k|^{2+\alpha}} \left(\int_{I_k} a \right)^{3+\alpha}.
\end{aligned} \tag{4.21}$$

Step 5. In case b) we also estimate the integral in (4.19). Similarly to Step 3, by Cutting Out Lemma we can make the estimate

$$\begin{aligned}
&\int_{\tau_n}^{\tau_{n+1}} a(t) \left(\min \left\{ \int_{\tau_n}^{t} \exp[-q(B(t) - B(s)) \, \mathrm{d}s]; \tau_{n+1} - t \right\} \right)^{2+\alpha} \mathrm{d}t \\
&\quad \ge \exp[-(2+\alpha)q\bar{b}_I] \sum_{p=0}^{P_n} \int_{\alpha_{p_n+p}}^{\beta_{p_n+p}} a(t) \left(\min \left\{ t - \alpha_{p_n+p}; \beta_{p_n+p} - t \right\} \right)^{2+\alpha} \mathrm{d}t \\
&\quad = c_{12} \sum_{p=0}^{P_n} \left(\int_{I_{p_n+p}} a \right)^{3+\alpha} \ge c_{12} \sum_{p=0}^{P_n} \frac{1}{1 + |I_{p_n+p}|^{2+\alpha}} \left(\int_{I_{p_n+p}} a \right)^{3+\alpha}
\end{aligned} \tag{4.22}$$

with a suitable defined constant c_{12}.

By the combination of (4.21) and (4.22) condition (2.15) implies the desired divergence (4.19). $\square$

Now we turn to proofs of results of type "exponent 2".

Lemma 4.7. *For arbitrary constant $c > 0$ the two conditions*

$$\int_0^\infty \int_0^t \exp\left[-c(B(t) - B(s))\right] \mathrm{d}s\, \mathrm{d}t = \infty, \tag{4.23}$$

$$\int_0^\infty \int_0^t \exp\left[-(B(t) - B(s))\right] \mathrm{d}s\, \mathrm{d}t = \infty \tag{4.24}$$

are equivalent.

Proof. We apply [9, Theorem 1.1]. Introduce the notation

$$B^{-1}(r) := \inf\left\{t \in \mathbb{R}_+ : \int_0^t b \geq r\right\} \qquad (r \in \mathbb{R}_+).$$

In the cited theorem we proved that condition (4.24) is equivalent to

$$\sum_{n=1}^{\infty} (B^{-1}(nd) - B^{-1}((n-1)d))^2 = \infty \tag{4.25}$$

for arbitrary $d > 0$. Define $b_c(t) := cb(t)$ and $B_c(t) := \int_0^t b_c = cB(t)$. Setting $d = 1$, from the cited theorem we obtain that

$$\begin{aligned} \int_0^\infty \int_0^t \exp\left[-\left(\int_0^t cb(u)\, \mathrm{d}u - \int_0^s cb(u)\, \mathrm{d}u\right)\right] \mathrm{d}s\, \mathrm{d}t \\ = \int_0^\infty \int_0^t \exp\left[-(B_c(t) - B_c(s))\right] \mathrm{d}s\, \mathrm{d}t = \infty \end{aligned} \tag{4.26}$$

is equivalent to

$$\sum_{n=1}^{\infty} (B_c^{-1}(n) - B_c^{-1}((n-1))^2 = \sum_{n=1}^{\infty} \left(B^{-1}\left(\frac{1}{c}n\right) - B^{-1}\left(\frac{1}{c}(n-1)\right)\right)^2 = \infty. \tag{4.27}$$

By a repeated application of the cited theorem now with $d = 1/c$, (4.26) and(4.27) yield the statement of the lemma. $\square$

Proof of Theorem 3.1. In [4, Theorem 2.2] we proved that the integral positivity and

$$\int_0^\infty \int_0^t \exp\left[-q(M)(B(t) - B(s))\right] \mathrm{d}s\, \mathrm{d}t = \infty$$

are sufficient for the asymptotic stability. By Lemma 4.7 it is enough to show that (3.3) implies (4.24). This obviously follows from the estimate

$$\int_0^\infty \int_0^t \exp\left[-(B(t) - B(s))\right] \mathrm{d}s\, \mathrm{d}t \geq \sum_{n=1}^{\infty} \int_{\alpha_n}^{\beta_n} \left(\int_{\alpha_n}^t \exp[-(B(t) - B(s))]\, \mathrm{d}s\right) \mathrm{d}t. \qquad \square$$

Proof of Corollary 3.2. We apply the estimate

$$\begin{aligned} J_n &:= \int_{\alpha_n}^{\beta_n} \left(\int_{\alpha_n}^{t} \exp[-(B(t)-B(s))]\,\mathrm{d}s \right) \mathrm{d}t \\ &\geq \int_{\alpha_n}^{\beta_n} \left(\int_{\alpha_n}^{t} \exp[-\bar{b}_n(t-s)]\,\mathrm{d}s \right) \mathrm{d}t = \frac{1}{\bar{b}_n}|I_n| - \frac{1}{\bar{b}_n^2}\Big(1-\exp[-\bar{b}_n|I_n|]\Big) \end{aligned}$$

to condition (3.3). □

Proof of Corollary 3.3. In Theorem 3.1 we use the estimate

$$J_n \geq \int_{\alpha_n}^{\beta_n} \left(\int_{\alpha_n}^{t} \exp[-(B(\beta_n)-B(\alpha_n))]\,\mathrm{d}s \right) \mathrm{d}t = \frac{1}{2}\exp[-B_n]|I_n|^2. \qquad \square$$

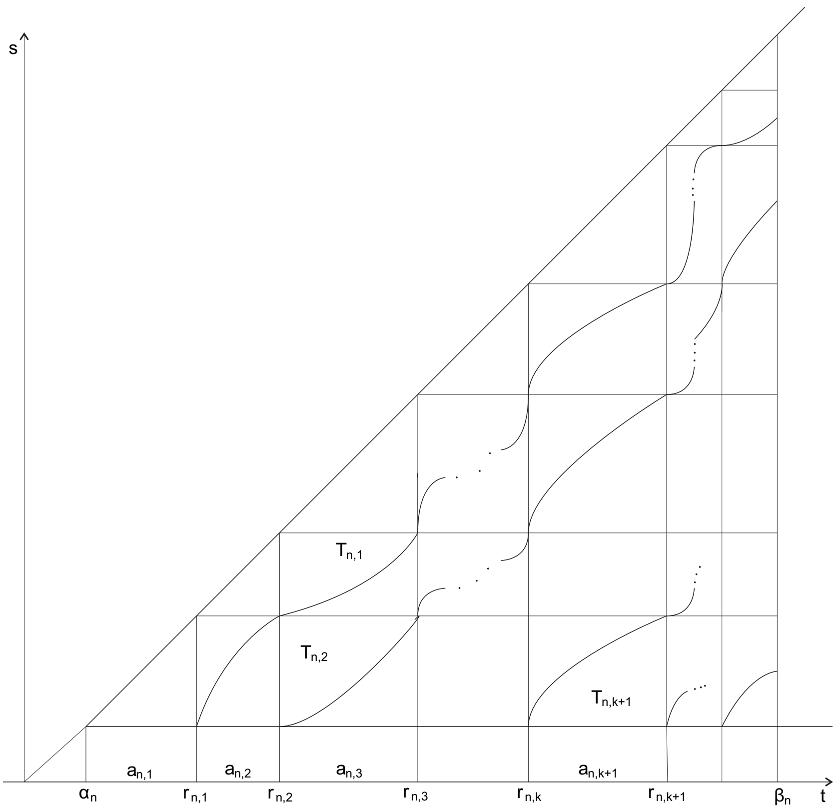

Figure 4.1: To the estimation of $\mathrm{mes}(T_{n,k})$

Proof of Theorem 3.4. Introduce the notations

$$\begin{gathered} T_{n,k}\{(t,s) \in [\alpha_n,\beta_n]\times[\alpha_n,\beta_n] : (k-1)d \leq B(t)-B(s) < kd\}, \\ a_{n,k} := r_{n,k} - r_{n,k-1} \qquad (k = 1,2,\dots). \end{gathered}$$

Then by Figure 4.1 we have

$$\begin{aligned} \mathrm{mes}(T_{n,k}) &\leq a_{n,1}(a_{n,k}+a_{n,k+1}) + a_{n,2}(a_{n,k+1}+a_{n,k+2}) + \dots \\ &\leq \left(a_{n,1}^2 + \frac{a_{n,k}^2 + a_{n,k+1}^2}{2}\right) + \left(a_{n,2}^2 + \frac{a_{n,k+1}^2 + a_{n,k+2}^2}{2}\right) + \dots \\ &\leq 2(a_{n,1}^2 + a_{n,2}^2 + \dots). \end{aligned} \tag{4.28}$$

On the other hand, obviously

$$\operatorname{mes}(T_{n,1}) \geq \frac{1}{2}(a_{n,1}^2 + a_{n,2}^2 + \dots). \tag{4.29}$$

From (4.29) and (4.28) we obtain the inequalities

$$\begin{aligned} L &:= \sum_{n=1}^{\infty} \int_{\alpha_n}^{\beta_n} \left(\int_{\alpha_n}^{t} \exp[-(B(t) - B(s))] \, \mathrm{d}s \right) \mathrm{d}t \\ &\geq \sum_{n=1}^{\infty} e^{-d} \operatorname{mes}(T_{n,1}) \geq \frac{1}{2} e^{-d} \sum_{n=1}^{\infty} \sum_{k=1}^{\infty} (r_{n,k} - r_{n,k-1})^2, \end{aligned}$$

and

$$\begin{aligned} L &\leq \sum_{n=1}^{\infty} \sum_{k=1}^{\infty} \exp[-(k-1)d] \operatorname{mes}(T_{n,k}) \\ &\leq 2 \sum_{n=1}^{\infty} \left(\left(\sum_{k=1}^{\infty} \exp[-(k-1)d] \right) \left(\sum_{k=1}^{\infty} a_{n,k}^2 \right) \right) \\ &\leq 2 \frac{e^d}{e^d - 1} \sum_{n=1}^{\infty} \left(\sum_{k=1}^{\infty} (r_{n,k} - r_{n,k-1})^2 \right), \end{aligned}$$

respectively, which complete the proof. □

References

[1] Á. Elbert, Stability of some difference equations, in: *Advances in difference equations (Veszprém, 1995)*, Gordon and Breach, Amsterdam, 1997, pp. 165–187. MR1636322; Zbl 0890.39010

[2] G. H. Hardy, J. E. Littlewood, G. Pólya, *Inequalities*, 2nd ed., University Press, Cambridge, 1952. MR0046395; Zbl 0047.05302

[3] P. Hartman, *Ordinary differential equations*, Birkhäuser, Boston, Mass., 1982. MR658490; Zbl 0476.34002

[4] L. Hatvani, Nonlinear oscillation with large damping, *Dynam. Systems Appl.* **1**(1992), 257–269. MR1182648; Zbl 0763.34042

[5] L. Hatvani, Integral conditions on the asymptotic stability for the damped linear oscillator with small damping, *Proc. Amer. Math. Soc.* **124**(1996), 415–422. https://doi.org/10.1090/S0002-9939-96-03266-2; MR1317039; Zbl 0844.34051

[6] L. Hatvani, On the damped harmonic oscillator with time dependent damping coefficient, *J. Dynam. Differential Equations* **30**(2018), 25–37. https://doi.org/10.1007/s10884-017-9597-z; MR3770291; Zbl 1387.34082

[7] L. Hatvani, Smith-type stability theorems for the damped linear oscillator, *Dynam. Systems Appl.* **27**(2018), 299–318. https://doi.org/10.12732/dsa.v27i2.6

[8] L. Hatvani, T. Krisztin, Necessary and sufficient conditions for intermittent stabilization of linear oscillators by large damping, *Differential Integral Equations* **10**(1997), 265–272. MR1424811; Zbl 0893.34045

[9] L. Hatvani, T. Krisztin, V. Totik, A necessary and sufficient condition for the asymptotic stability of the damped oscillator, *J. Differential Equations* **119**(1995), 209–223. https://doi.org/10.1006/jdeq.1995.1087; MR1334491; Zbl 0831.34052

[10] L. Hatvani, V. Totik, Asymptotic stability of the equilibrium of the damped oscillator, *Differential Integral Equations* **6**(1993), 835–848. MR1222304; Zbl 0777.34036

[11] P. Pucci, J. Serrin, Precise damping conditions for global asymptotic stability for nonlinear second order systems, *Acta Math.* **170**(1993), 275–307. https://doi.org/10.1007/BF02392788; MR1226530; Zbl 0797.34059

[12] P. Pucci, J. Serrin, Asymptotic stability for intermittently controlled nonlinear oscillators, *SIAM J. Math. Anal.* **25**(1994), 815–835. https://doi.org/10.1137/S0036141092240679; MR1271312; Zbl 0809.34067

[13] R. A. Smith, Asymptotic stability of $x'' + a(t)x' + x = 0$, *Quart. J. Math. Oxford Ser. (2)* **12**(1961), 123–126. https://doi.org/10.1093/qmath/12.1.123; MR0124582; Zbl 0103.05604